9.8 Power Series:

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ is called

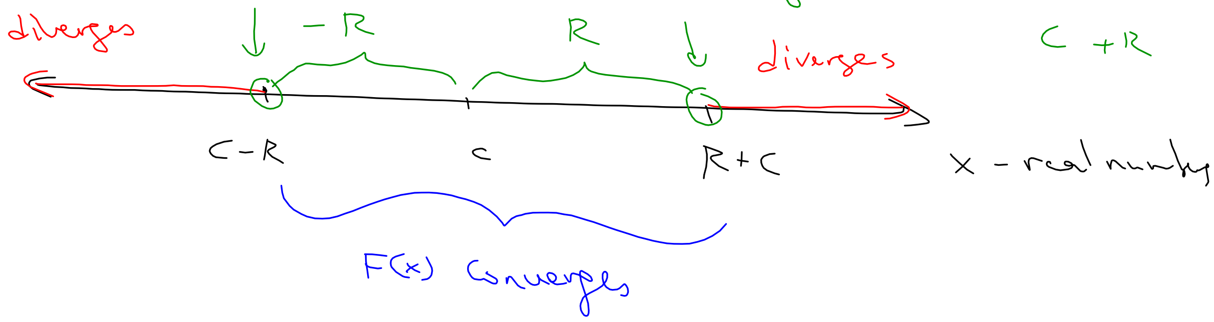
a power series centered at c.

$F(x)$ converges over an interval $(c-R,\ c+R)$ where R is called the radius of convergence

$$F(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + \cdots$$

Note: $F(x)$ converges at its center c, always!
since $F(c) = a_0$ (when $x = c$)

$F(x)$ may or may not converge at $x = c-R$ or $c+R$



Eg: Where does $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ie, find an interval where the values of x make $F(x)$ convergent!

Solution:

n^{th} Root Test Let

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{where } a_n = \frac{x^n}{2^n} = \left(\frac{x}{2}\right)^n$$

$$\text{So } L = \lim_{n\to\infty}\left(\left|\frac{x}{2}\right|\right) = \left|\frac{x}{2}\right|$$

$F(x)$ converges if $L < 1 \Longrightarrow \left|\frac{x}{2}\right| < 1$

$$\Longrightarrow |x| < 2$$

$$\Longrightarrow -2 < x < 2$$

Endpoints: When $x = -2$

$$F(-2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\cdots$$

The values alternate between "0" and "1" So diverges (or by the n^{th} test of divergence test!)

$$F(2) = \sum_{n=0}^{\infty} \frac{(2)^n}{2^n} = \sum_{n=0}^{\infty} 1$$

(diverges by n^{th} test of divergence)

So, the interval of convergence of $F(x)$ is

$$x \in \boxed{(-2, +2)}$$

Eg: $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n} (x-5)^n$

Find the interval of convergence of $F(x)$

Solution:

Ratio Test. Let $a_n = \frac{(x-5)^n}{4^n n}$,

So $a_{n+1} = \frac{(x-5)^n (x-5)}{4^n \cdot 4 \cdot (n+1)}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-5)^n (x-5)}{4^n 4 (n+1)} \cdot \frac{4^n n}{(x-5)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-5)\,n}{4(n+1)} \right|$$

$$= \frac{|x-5|}{4} \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \to 1$$

So $L = \frac{|x-5|}{4}$ and $F(x)$ converges if

$L < 1 \implies \frac{|x-5|}{4} < 1 \implies |x-5| < 4$

$$-4 < x-5 < +4$$
$$+5 \quad\quad +5 \quad\quad +5$$
$$+1 < x < 9$$

Endpoints: $x = 1$

$* \quad F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n}\right)$

Alternating harmonic series converges (conditionally),

When $x = 9$

$F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Converges as in the previous case!

Therefore $F(x)$ converges over

$$\boxed{[1, 9]}$$

Ex: $F(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $n! = 1 \cdot 2 \cdot 3 \cdots n$

Find the interval of convergence of $F(x)$!

Solution

$$L = \lim_{n\to\infty} \left| \frac{x^{2n} \cdot x^2}{2n!(2n+2)} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{2n+2} \right| = |x^2| \lim_{n\to\infty} \left| \frac{1}{2n+2} \right|$$

$= |x^2| 0 = 0$ So $L < 1$ always for all values of x, in which case $R = \infty$

Therefore $F(x)$ converges over $(-\infty, +\infty)$

Ex: $F(x) = \sum_{n=0}^{\infty} n!\, x^n$, Find the interval of convergence of $F(x)$.

Solution:

$$l = \lim_{n\to\infty} \left| \frac{n!(n+1)\,x^n \cdot x}{n!\, x^n} \right|$$ where $a_n = n!\, x^n$

$a_{n+1} = (n+1)!\, x^{n+1} = n!(n+1)\, x^n \cdot x$

$$= \lim_{n\to\infty} \left| (n+1)x \right| = |x| \lim_{n\to\infty} |(n+1)|$$

$L > 1$ for all $|x| > 0$; ie $|x| \lim_{n\to\infty} |n+1| = \infty$

$F(x)$ converges only at $\boxed{x=0}$

(9.9) Representation of functions by power series

Recall $\sum ar^n \longrightarrow \frac{a}{1-r}$, $|r|<1$

If we let $r=x$, $\sum ax^n = \frac{a}{1-x}$, $|x|<1$

So $\frac{1}{1-\underbrace{x}_{r}} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$

Eg: Find a power series with center $c=0$ for $f(x)=\frac{1}{2+x^2}$

Solution: $f(x) = \frac{1}{2\left(1+\frac{x^2}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x^2}{2}\right)\right]} = \frac{\frac{1}{2}}{1-\left[\frac{-x^2}{2}\right]}$

$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$

$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}}$

for $\left|\frac{-x^2}{2}\right|<1 \rightarrow |x|<\sqrt{2}$

Ex: Find a power series of $\frac{4}{2+x} = f(x)$

Solution: $f(x) = \frac{4}{2\left(1+\frac{x}{2}\right)} = \frac{2}{1-\left(\frac{-x}{2}\right)} = \sum_{n=0}^{\infty} 2\left(\frac{-x}{2}\right)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n-1}}$

with $|x|<2$

Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ converges

with $x \in (c-R, c+R)$, then

$$F'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

and $\int F(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + A$ where A is a constant

and each series converges over the same interval

Eg: Find a power series of $f(x) = \frac{1}{(1-x)^2}$

Solution

Observe $g(x) = \frac{1}{1-x}$, $g'(x) = \frac{1}{(1-x)^2} = f(x)$

So, $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$

Now $f(x) = g'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} n x^{n-1}$

Thus $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = 0 + 1 + 2x^1 + 3x^2 + \dots$

Ex: Find the power series of $f(x) = \ln x$

Solution: Let $g(x) = \ln x \rightarrow g'(x) = \frac{1}{x}$

Note: $\frac{1}{x} = \frac{1}{1-(-x+1)} = \sum_{n=0}^{\infty} (-x+1)^n$, $|-x+1| < 1$

$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ center 1

Converges with $x \in (0, 2)$

So $g(x) = \int \frac{1}{x} dx$

$= \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n dx = \sum \frac{(-1)^n}{n+1} (x-1)^{n+1} + A$

where A is a constant.

At the center $c = 1$, $\ln(1) = 0 \rightarrow A = 0$

So $f(x) = \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$

$= 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$

Ex: Find a power series centered at 0 of

$f(x) = \arctan x$

Solution

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\Longrightarrow \text{Arctan}\, x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \sum \frac{(-1)^n}{2n+1} x^{2n+1} + A$$

At the center $x = 0 \longrightarrow \text{Arctan}(0) = 0 = 0 + A \rightarrow A = 0$

Therefore $\text{Arctan}\, x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\frac{\pi}{4} = \text{Actan}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$